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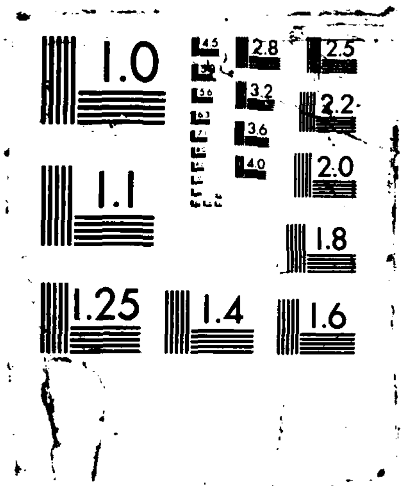
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CMS Technical Summary Report #88-11

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Gerardo A. Ache

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ABSTRACT

At re-entrant corners the pressure has a singularity for incompressible viscous flow. In fluid flow computations there are geometries that have re-entrant corners, and for which it is needed to provide an appropriate value for the pressure at such a corner when a finite difference method dealing with the primitive formulation is used. In this paper we address the problem of finding an efficient strategy for computing pressure values at a re-entrant corner which applied to Strikwerda's second-order numerical method for solving the Stokes and Navier-Stokes equations. The pressure at the corner is regarded as a double valued function. Also we examine Moffatt's solution for the Stokes' problem near a step where the pressure becomes unbounded as the re-entrant corner is approached. We show that this strategy models very well the pressure singularity making the computation more amenable and efficient.

AMS(MOS) Subject Classifications: 65C20, 76-08, 76D07
Key Words: Navier-Stokes, re-entrant corner

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NUMERICAL TREATMENT OF THE PRESSURE SINGULARITY

AT A RE-ENTRANT CORNER

Gerardo A. Ache*

1. Introduction

For plane steady incompressible viscous flow, it is known that in geometries with corners the flow close to the corner will be dominated by the Stokes' solution and the governing equation for the stream function will be the biharmonic equation [8] i.e.

$$\nabla^4 \psi = 0 \quad , \tag{1.1a}$$

with ψ satisfying the non-slip boundary conditions

$$\psi = \psi_0 \ , \ \frac{\partial \psi}{\partial n} = 0 \quad . \tag{1.1b}$$

As pointed out in [2], [7] and [8], corner flows lead to a singularity in the vorticity and the pressure. As examples of such problems having geometries with corner, that one sees often in applications, are flow in a cavity and flow in a channel with a step (or a pipe with a sudden enlargement of the cross section). For flow in a cavity, in most of the finite difference schemes, the singularity for the pressure or the vorticity may be avoided if it is assumed that the presence of such singularities affects the solution only in a small neighborhood of the corner. We find a different situation for flow near a step, that is a re-entrant corner. Even if the singularity, in the pressure or vorticity, is assumed to be

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negligible it is needed to provide, appropriate values for the pressure or vorticity to evaluate pressure or vorticity derivatives at grid points near the corners. An excellent survey about vorticity strategies at a re-entrant corner can be found in [5].

In this paper, we address the problem of providing appropriate pressure values, at a re-entrant corner, when a finite difference scheme for the primitive formulation of the Stokes or Navier-Stokes equation is used. As an example of such a scheme we examine [10]. We show that using the approach adopted in this paper, we are able to compute numerical solutions for the Stokes equations, in domains with re-entrant corners, for which the numerical pressure accurately models the pressure singularity.

2. The Fluid Flow Problem Near a Sharp Corner

In this section we examine Moffatt's solution [8] for plane flow near a corner. The stream function ψ is defined in polar coordinates by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad , \quad v = -\frac{\partial \psi}{\partial r} \quad , \tag{2.1}$$

where if the corner is at (x_0, y_0) then $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$. We assume that the corner is given by $-\alpha \le \theta \le \alpha$, and the governing equation, in polar coordinates, is given by

$$\begin{aligned} \nabla^4\psi = \frac{\partial^4\psi}{\partial r^4} + \frac{2}{r}\frac{\partial^3\psi}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^3}\frac{\partial\psi}{\partial r} + \frac{2}{r^2}\frac{\partial^4\psi}{\partial r^2\partial\theta^2} \\ -\frac{2}{r^3}\frac{\partial^3\psi}{\partial r\partial\theta^2} + \frac{4}{r^4}\frac{\partial^2\psi}{\partial\theta^2} + \frac{1}{r^4}\frac{\partial^4\psi}{\partial\theta^4} = 0. \end{aligned} \tag{2.2}$$

In [7] Lugt and Schwiderski have shown that the equation (2.2) admits separable solutions of the form,

$$\psi(r,\theta) = r^\lambda f_\lambda(\theta) \quad , \tag{2.3}$$

where

$$f_\lambda(\theta) = A_\lambda \sin \lambda\theta + B_\lambda \cos \lambda\theta + C_\lambda \sin(\lambda - 2)\theta + D_\lambda \cos(\lambda - 2)\theta \quad , \tag{2.4a}$$

for $\lambda \neq 1$ and ,

$$f_1(\theta) = A_1 \sin 2\theta + B_1 \cos 2\theta + C_1\theta + D_1 \quad . \tag{2.4b}$$

Thus the general solution to (2.2) can be written in a series of the form ,

$$\psi = \psi_0 + \sum_{n=1}^{\infty} H_n r^{\lambda_n} f_{\lambda_n}(\theta) \quad , \tag{2.5}$$

with the eigenvalues λ_n satisfying

$$0 \leq Re(\lambda_1) \leq Re(\lambda_2) \leq \cdots \quad .$$

In order that ψ satisfies the boundary conditions (1.1b) it is required that λ_n satisfies an eigenvalue equation. For symmetric slow motions ($B_\lambda = D_\lambda = 0$) this equation becomes, (for $\lambda_n \neq 1$)

$$\sin 2\alpha(\lambda_n - 1) = (\lambda_n - 1) \sin 2\alpha \quad , \tag{2.6a}$$

while for skew symmetric motions ($A_\lambda = C_\lambda = 0$) ,

$$\sin 2\alpha(\lambda_n - 1) = -(\lambda_n - 1) \sin 2\alpha \quad . \tag{2.6b}$$

The pressure P_λ can be recovered using the stream function ψ and the Stokes equations. Neglecting an arbitrary constant one arrives at (for $\lambda \neq 1$)

$$P_\lambda(r,\theta) = -4\lambda r^{\lambda-2} C_\lambda \cos(\lambda - 2)\theta \quad , \tag{2.7a}$$

for symmetric motions, and

$$P_\lambda(r,\theta) = 4\lambda r^{\lambda-2} D_\lambda \sin(\lambda - 2)\theta \quad , \tag{2.7b}$$

for skew symmetric motions. In general the final pressure is given by the superposition of symmetric and skew symmetric motions, for all λ satisfying (2.6a) or (2.6b). For flow near a step, where $2\alpha = 3\pi/2$, we can combine (2.6a), (2.6b) so that,

$$(\lambda_n - 1)(-1)^{n+1} = \sin(\lambda_n - 1)\frac{3\pi}{2} \quad , \tag{2.8}$$

where for n odd the λ_n's are associated to symmetric motions and for n even with skew symmetric motions. In Table I we list the first four eigenvalues for this problem.

TABLE I

Symmetric and Skew-symmetric Eigenvalues

n	Eigenvalues
1	1.54448
2	1.90853
3	2.62926+0.23125i
4	3.30133+0.31584i

The final pressure for the Stokesian flow, near a re-entrant corner, can be written as

$$p(r,\theta) = \sum_n H_n P_{\lambda_n}(r,\theta) \quad , \tag{2.9}$$

where the coefficients $C_{\lambda_n}, D_{\lambda_n}$ in P_{λ_n} are determined using the boundary conditions. One sees from (2.7a), (2.9) and Table I that the pressure becomes unbounded as r approaches the re-entrant corner and the leading term is approximately of the order $O(r^{-0.456})$.

For the fully non-linear two dimensional Navier-Stokes equations it is possible to analyze the solution near a corner using the technique described in [2]. This approach is based on a semi-analytic perturbation analysis which is valid near a corner for small

Reynolds numbers. The use of this technique may be useful to study the nature of the singularity of the pressure near a re-entrant corner for small values of R .

3. Numerical Treatment of the Pressure Singularity at a Re-Entrant Corner

Although the pressure at a re-entrant corner may be unbounded, a finite value is needed for the finite difference scheme at such a corner to evaluate the pressure gradient at adjacent grid points. A similar situation is encountered when the stream function-vorticity formulation is used since the vorticity is unbounded at the re-entrant corner and yet a finite value is needed to evaluate first and second derivatives at adjacent grid points. Some corner strategies for the vorticity when a finite difference approach is used have been discussed in [4], [5], [6], [9] .

In this paper we provide a finite difference strategy for the pressure applied to the Stokes and Navier-Stokes equations. The numerical method that we used for that purpose is based on a second-order finite difference scheme to solve the Stokes and Navier-Stokes equations [10], [11]. This method deals with primitive variables formulation, i.e., it uses the velocity and the pressure as dependent variables. Then the laplacian and convection terms, given in conservative form, are discretized using standard centered difference formulas, while the pressure gradient and the continuity equation are discretized using regularized centered difference formulas [10], e.g.

$$\frac{\partial p}{\partial x} \approx \delta_{x0}p - \frac{1}{6}h^2\delta_{x-}\delta_{x+}^2 p \quad , \tag{3.1a}$$

$$\frac{\partial p}{\partial y} \approx \delta_{y0}p - \frac{1}{6}h^2\delta_{y-}\delta_{y+}^2 p \quad , \tag{3.1b}$$

$$\frac{\partial u_1}{\partial x} \approx \delta_{x0}u_1 - \frac{1}{6}h^2\delta_{x+}\delta_{x-}^2 u_1 \quad , \tag{3.1c}$$

$$\frac{\partial u_2}{\partial y} \approx \delta_{y0} u_2 - \frac{1}{6} h^2 \delta_{y+} \delta_{y-}^2 u_2 \quad , \tag{3.1d}$$

where $(\delta_{x0}, \delta_{y0})$, $(\delta_{x-}, \delta_{y-})$, $(\delta_{x+}, \delta_{y+})$ are centered, backward, and forward differences in the x and y direction, respectively. Formulas (3.1) are used in the pressure gradient and the continuity equation. To determine the pressure on the boundaries cubic interpolation formulas are used, e.g. along a boundary given by i or j equal to zero,

$$p_{0,j} = 3(p_{1,j} - p_{2,j}) - p_{3,j} \quad , \tag{3.2a}$$

$$p_{i,0} = 3(p_{i,1} - p_{i,2}) - p_{i,3} \quad . \tag{3.2b}$$

To solve the resulting finite difference equations an extended S.O.R. iterative procedure [11] was used. This numerical method is very efficient and accurate. It has been used to compute flow in a spinning and coning cylinder [12].

Since the numerical scheme requires values of the pressure at the solid wall and since these values are computed using extrapolation formulas, we may compute the pressure at the corner in two different ways using such extrapolation formulas. For each of the two directions we let $P_{c,x}$ and $P_{c,y}$ be the pressure value determined by extrapolation in the x direction and y direction. Assuming that the re-entrant corner is located at (x_0, y_0), we have

$$P_{c,x} := p(x_0, y_0)$$

$$= 3(p(x_0 + h_x, y_0) - p(x_0 + 2h_x, y_0)) + p(x_0 + 3h_x, y_0) \quad , \tag{3.3a}$$

and

$$P_{c,y} := p(x_0, y_0)$$

$$= 3(p(x_0, y_0 + h_y) - p(x_0, y_0 + 2h_y)) + p(x_0, y_0 + 3h_y) \quad . \tag{3.3b}$$

Here h_x, h_y are the grid sizes in the x and y directions respectively. Notice that since these values of $P_{c,x}$ and $P_{c,y}$ do not need to be equal the pressure at a re-entrant corner may be regarded as a double valued function. Therefore to evaluate the pressure gradient at adjacent grid points we use $P_{c,x}$ to compute the approximation to $\partial p/\partial x$ at $(x_0 + h_x, y_0)$ and $P_{c,y}$ to compute the approximation to $\partial p/\partial y$ at $(x_0, y_0 + h_y)$. This strategy works well in computation at small and moderate Reynolds numbers. This approach applies also to axi-symmetric and three-dimensional computations which include flow near a re-entrant corner.

It is interesting to note that the singular nature of the pressure at a re-entrant corner has not been appreciated by computational fluid dynamists. For example, Roache [9] suggests that if two values are obtained for the pressure at such a corner, then their difference should be used as a measure of truncation error. He claims that a non-single valued pressure is non-physical. Computationally, for the Stoke problem, the difference in the two pressure values given by (3.3a) and (3.3b) tends to be unbounded as the mesh is refined (see Table III).

4. Check of Accuracy

As we said in §3, the finite difference method we used to solve the Navier-Stokes equations is second-order accurate. The fact that the pressure possesses a singularity at the re-entrant corner may affect the accuracy of the scheme near the step. To check the accuracy we used values of the stress function at the wall, and at different positions downstream, since it represents a sensitive quantity in the computation. The stress at the

wall is defined by

$$\tau = -p + \frac{\partial v}{\partial y} \quad , \tag{4.1}$$

where v is the transversal velocity. Let τ^u and τ^l be the stresses at the upper and lower wall of the stepped channel, respectively, (see Figure 1) then in order to eliminate the pressure constant, we define the value $\Delta\tau$ by

$$\Delta\tau = \tau^u - \tau^l \quad . \tag{4.2}$$

The check consists of taking three different values $\Delta\tau_{h_1}$, $\Delta\tau_{h_2}$, $\Delta\tau_{h_3}$ corresponding to the computed values of (4.2) using different grid sizes h_1, h_2, h_3 , where $h_k = \frac{1}{2}h_{k-1}$ and $k = 2,3$. Then we assume that $\Delta\tau$ can be represented as

$$\Delta\tau_h = \Delta\tau + Ch^m + O(h^q) \quad , \tag{4.3}$$

where $q > m$ and C is a constant. By neglecting smaller order terms in (4.3) we have that the order m, is given by

$$m = \log\left(\frac{\Delta\tau_{h_2} - \Delta\tau_{h_1}}{\Delta\tau_{h_3} - \Delta\tau_{h_2}}\right) \Big/ \log\ 2 \quad . \tag{4.4}$$

We made the computations in the domain with the inflow boundary at $x = -1$ and the outflow boundary at $x = 1$, where the specification of Poiseuille flow was used at inlet and outlet. The test positions were taken at $x = \frac{1}{4}, x = \frac{1}{2}$, and $x = \frac{3}{4}$. We used two different initial grids of size $h_1 = 1/12$ and $h_1 = 1/16$, respectively. The resulting accuracy is given in Table II.

TABLE II

Accuracy for the Stokes' problem in the stepped channel

h_1	x	m
1/12	1/4	.977
	1/2	1.408
	3/4	4.030
1/16	1/4	1.187
	1/2	1.594
	3/4	3.749

We are not aware of any previous work involving the computation of incompressible fluid flow in domains with re-entrant corners in which a similar accuracy check had been carried out. This check also demonstrates that away from the step the numerical scheme retains the second order accuracy. The fact that the accuracy is so high at x equal to 3/4 is probably due to the simple nature of v and p in Poiseuille flow and also due to the third-order accurate difference formulas for the divergence of the velocity and the pressure gradient.

5. Results

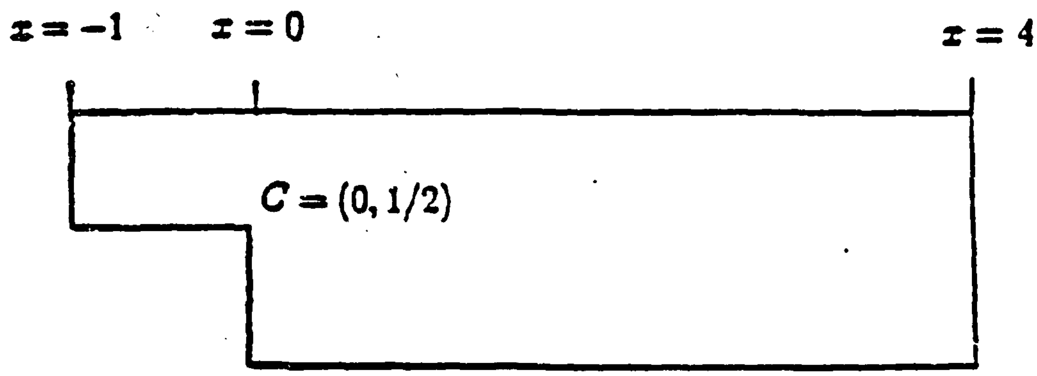

Figure 1: A Channel with a step.

We computed flow in a channel and in an axi-symmetric pipe, both geometries have re-entrant corners (Figure 1), the Reynolds numbers were in the range $0 \leq R \leq 50$, we use the pressure corner strategy discussed above. The results include graphics of the pressure along the step, (i.e., $x = 0$, $0 \leq y \leq 1$) and also values of the pressure gap which is $[p] := | P_{c,x} - P_{c,y} |$. All the computations were done on the VAX 11/780 at the Mathematics Research Center at the University of Wisconsin-Madison. To check the self consistency of the approach we introduce the following device. Since near the corner the pressure behaves like $p \sim A_1 r^{-0.456} + O(r^{-0.091})$, with A_1 a constant depending on the angle θ , then the gap $[p]$ should behave, approximately, as

$$h^{0.456}[p] \doteq \bar{C} + O(h^{0.364}) \quad , \tag{5.1}$$

with $\bar{C}$ a constant and h the grid size. Table III shows the values of of the pressure gap

$[p]$ and the values for $h^{0.456}[p]$, at ten different grid sizes using the same domain and boundary conditions as in §4.

TABLE III

Pressure gap for different grid sizes

h	$[p]$	$h^{0.456}[p]$
$\frac{1}{12}$	65.3894	21.08119
$\frac{1}{16}$	71.5178	20.22504
$\frac{1}{20}$	77.1634	19.71243
$\frac{1}{24}$	82.4458	19.38327
$\frac{1}{28}$	87.4137	19.15760
$\frac{1}{32}$	92.0902	18.99143
$\frac{1}{36}$	96.5233	18.86576
$\frac{1}{40}$	100.7433	18.76781
$\frac{1}{44}$	104.7752	18.68960
$\frac{1}{48}$	108.6414	18.62613

The agreement with the prediction (5.1) becomes better as the mesh size is refined, however since the lower order term decays very slowly the third column in Table III tends slowly to a constant for very fine meshs.

Figures 2.a and 2.b show the pressure at the step, $R = 0$, for two dimensional flow in a channel and axi-symmetric flow in a pipe using grids of 12×12, 24×24 and 48×48 points, respectively. Note that for the pipe flow problem the Stokesian pressure (Figure 2.b) presents the same type of behavior as for the channel flow problem, i.e., that the

pressure gap becomes unbounded as the mesh size becomes finer. Figures 3.a and 3.b show the pressure at the step for R greater than zero, the computations were performed at Reynolds numbers equal to 30, 40 and 50, for the channel flow problem and at Reynolds numbers equal to 10, 20 and 30 for the pipe flow problem. From Figures 3.a and 3.b, it can be observed that as the Reynolds number becomes larger the pressure gap tends to be smaller. It seems to be that the assumption that near the corner the solution of the Navier-Stokes equations can be expanded in terms of Moffatt's eigenfunctions, is only valid for small Reynolds numbers.

6. Conclusion

We have presented a method of treating the pressure at a re-entrant corner, this strategy is motivated by the need for obtaining pressure values at the wall. The resulting pressure value at the corner may be regarded as double valued. These values of the pressure at the re-entrant corner are used to evaluate the pressure gradient at adjacent grid points. This strategy gives results which are consistent with decreasing mesh length and also can be used to treat axi-symmetric and three-dimensional viscous corner flow without introducing any modification in the nature of the algorithm. We used as test problems channels and axi-symmetric geometries with re-entrant corners and computations were performed at low and moderate Reynolds numbers. This way to treat the pressure at a re-entrant corner seems to work the best, among a few other strategies we used, applied to the numerical scheme in [10]. The numerical results show that the pressure singularity can be modeled very accurately, making the computation more efficient.

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References

[1] G. A. Ache, "Incompressible Viscous Steady Flow in Channel and Pipes", Ph.D. dissertation, the University of Wisconsin-Madison (1985).

[2] M. M. Gupta, R. P. Manohar and B. Noble, Computers and Fluids **9** (1981), 379.

[3] C. Hancock, Q. J. Mech. Appl. Math. **37** (1984), 113.

[4] H. Holstein and D. J. Paddon, J. Non-Newtonian Fluid Mech. **8** (1981), 81.

[5] H. Holstein and D. J. Paddon, A Finite Difference Strategy for Re-Entrant, in "Numerical Methods for Fluid Dynamics" (Baines and Morton, eds.) Academic Press (1983), 342.

[6] M. A. Kelmanson, Computers and Fluids **11** (1983), 307.

[7] H. J. Lugt and E. W. Schwiderski, Proc. Roy. Soc. Lond. **A 285** (1965), 382.

[8] M. K. Moffatt, J. Fluid Mech. **18** (1964), 1.

[9] P. J. Roache, "Computational Fluid Dynamics", Hermosa Pub., Albuquerque (1972).

[10] J. C. Strikwerda, SIAM J. Sci. Statist. Comp. **5** (1984), 56.

[11] J. C. Strikwerda, SIAM J. Numer. Anal. **21** (1984), 447.

[12] J. C. Strikwerda and Y. M. Nagel, to appear in J. Comput. Phys.

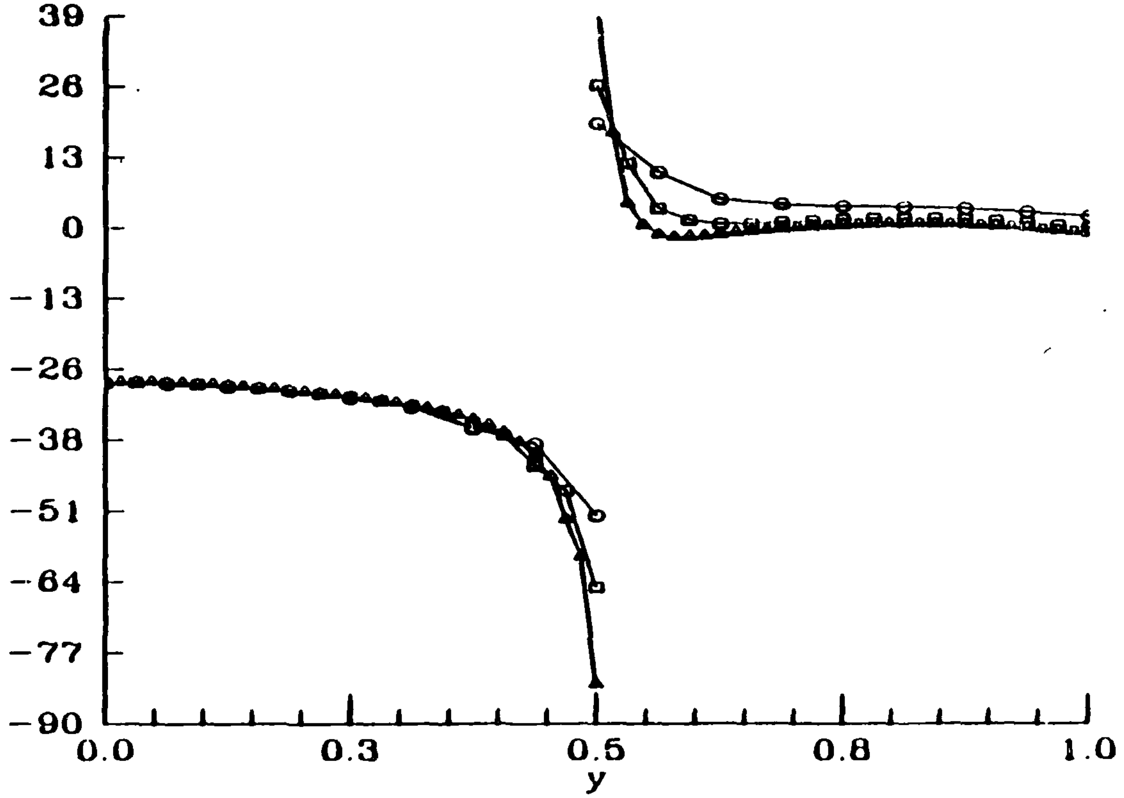

FIGURE 2.a : Channel flow problem. Pressure at the step ($x = 0$)

for $R = 0$ $h = 1/12,\ 1/24,\ 1/48$. ($\circ, \square, \triangle$).

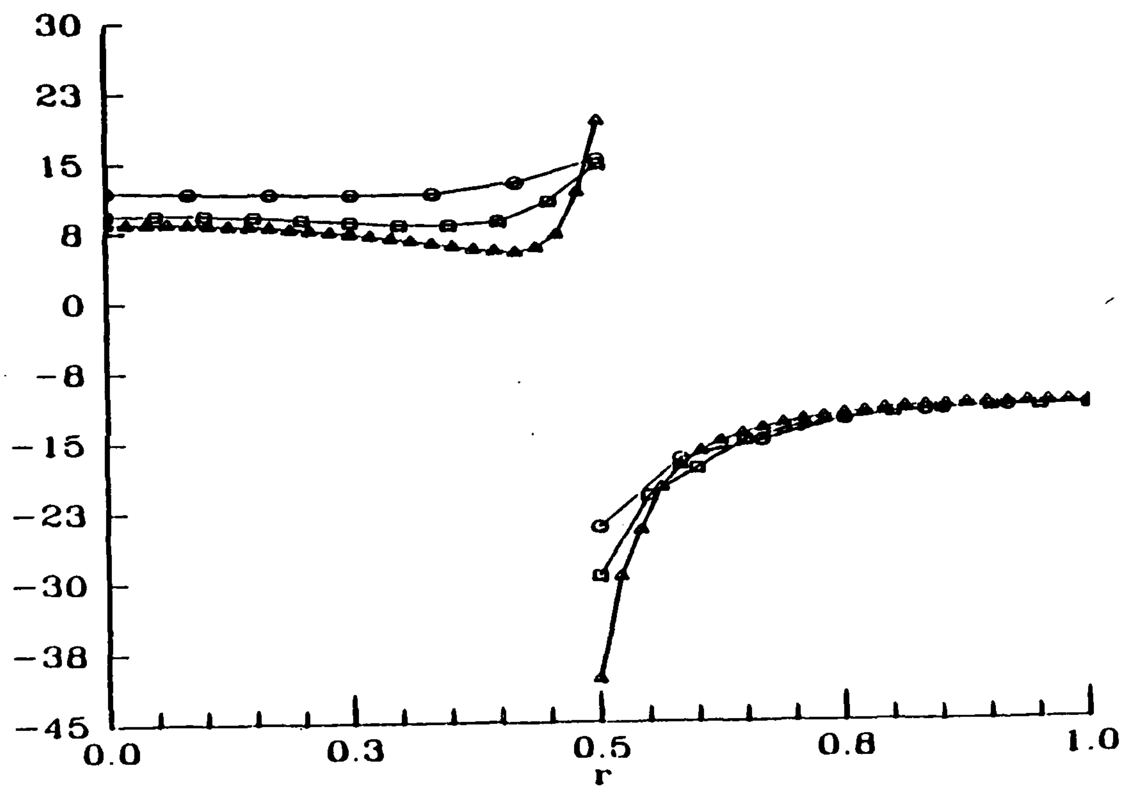

FIGURE 2.b : Pipe flow problem. Pressure at the step ($z = 0$)

for $R = 0$ $h = 1/12, 1/24, 1/48$. ($\circ, \square, \triangle$).

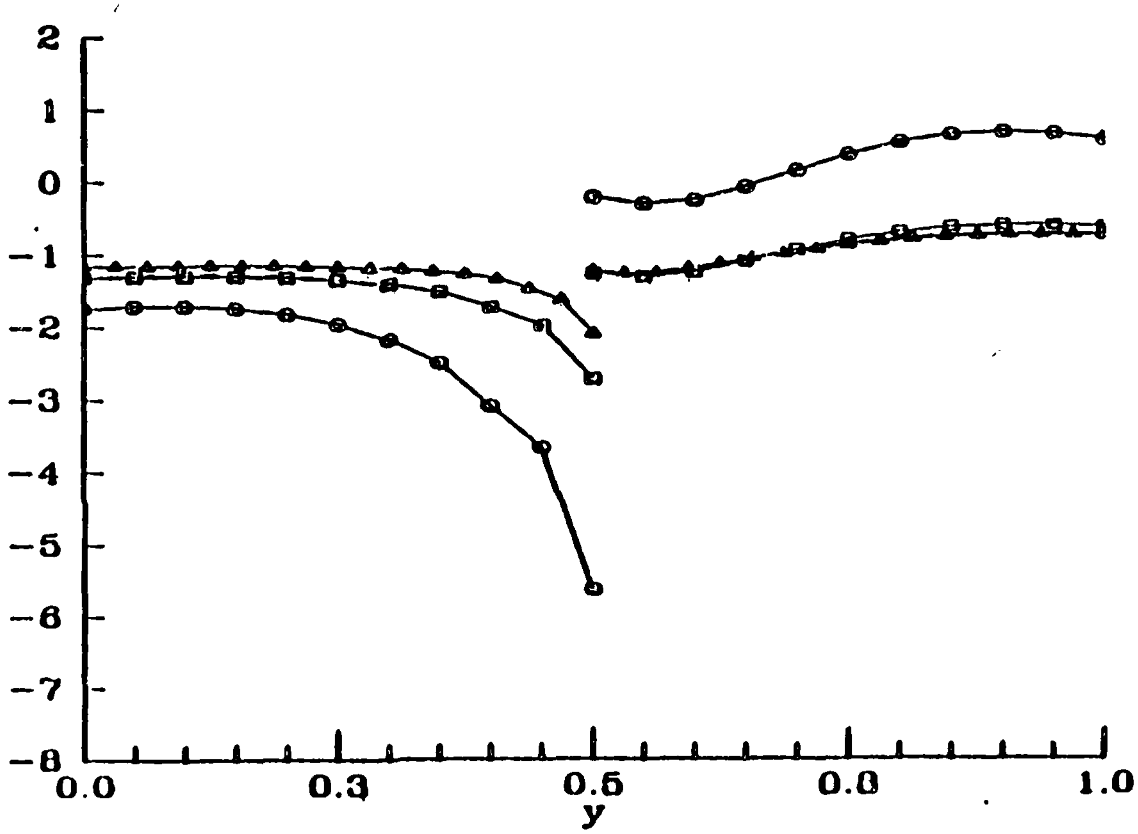

FIGURE 3.a : Channel flow problem. Pressure at the step ($x = 0$)

$R = 10, 30, 50$ ($\circ, \square, \triangle$).

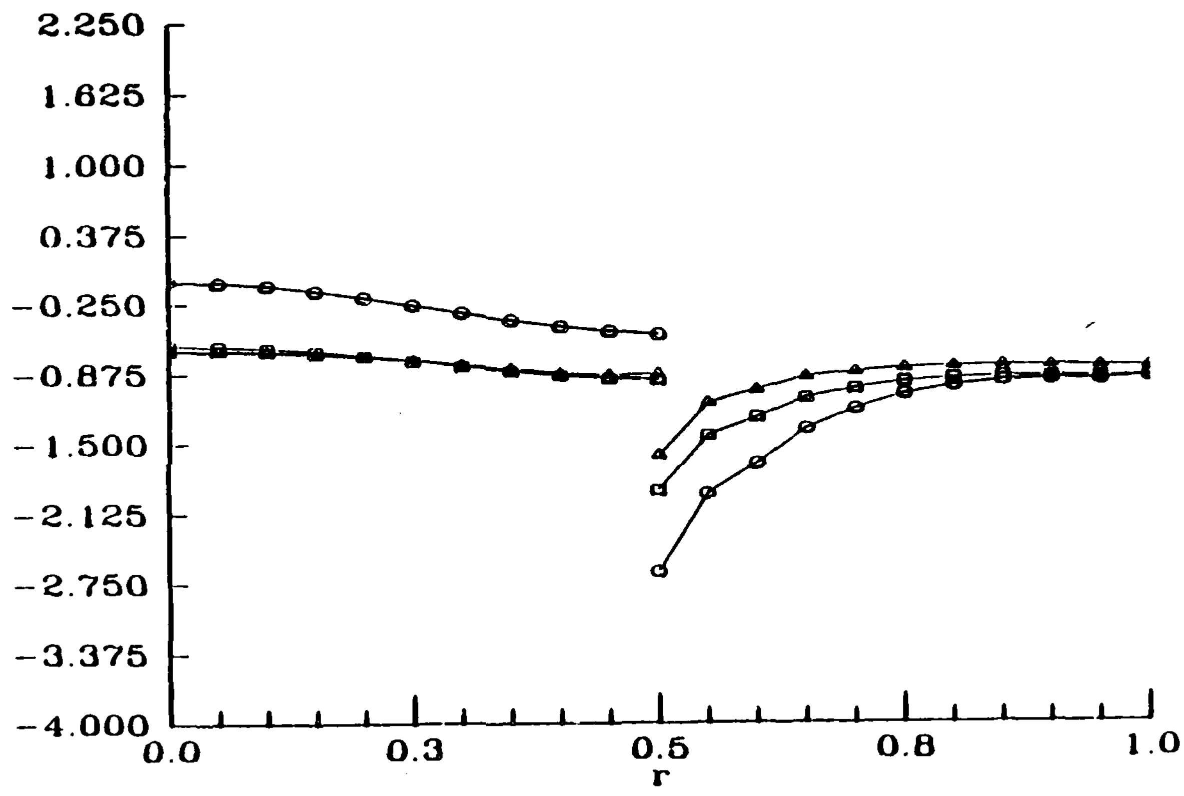

FIGURE 3.b : Pipe flow problem. Pressure at the step ($z = 0$)

for $R = 10, 20, 30$ ($\circ, \square, \triangle$).

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